INTERACTION OF AN ELASTIC BOUNDARY WITH A VISCOUS SUBLAYER OF A TURBULENT BOUNDARY LAYER

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The boundary region of a turbulent boundary layer contributes greatly to the drag. Intense turbulence is generated in this region. Below we investigate the interaction of an elastic boundary with a viscous sublayer for a decrease in the Reynolds stresses, and for a corresponding decrease in the drag. It does not seem possible to investigate the general case. Therefore, the problem is solved within the framework of the limitations made by Sternberg [1] for the theory of a viscous sublayer in a turbulent flow near a solid smooth wall.

We consider the system consisting of the three equations of motion and the equation of continuity

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},\quad \frac{\partial v}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\nu\frac{\partial^2 v}{\partial y^2}$$
$$\frac{\partial w}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\nu\frac{\partial^2 w}{\partial y^2},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

Here p is the pulsed pressure; ν is the kinematic coefficient of viscosity; ρ is the fluid density; and u,v, and w are the components of the pulsed velocity in the direction of the coordinate axes.

We assume that it is possible to expand the pulsations in a Fourier series and carry out operations on the separate terms.

The variation in velocity pulsations across the sublayer (along y) is most significant in comparison with their variation along the flow (along x) or in the z direction; therefore, the latter two are not considered. We seek a solution in the sublayer in the form

$$\begin{aligned} u &= \mathrm{Re}\,\{h(y)\exp[i(k_x x+k_z z-\beta t)]\}\\ v &= \mathrm{Re}\,\{g(y)\exp[i(k_x x+k_z z-\beta t)]\}\\ w &= \mathrm{Re}\,\{k(y)\exp[i(k_x x+k_z z-\beta t)]\}\\ k_x &= 2\pi/\lambda_x,\quad k_z=2\pi/\lambda_z,\quad \beta=2\pi f \end{aligned} \tag{2}$$

Here h(y), g(y), and k(y) are the unknown complex functions, β is the angular frequency, and k_x is a wave number such that the velocity of perturbation translation in the flow direction is

$$U_w=\beta/k_x \tag{3}$$

The pulsed pressure is

$$p=\mathrm{Re}\,\{p_\vartheta\exp[i(k_x x+k_z z-\beta t)]\} \tag{4}$$

Here p_ϑ is also a complex quantity.

The boundary conditions are written as follows. The sticking condition for the fluid must be satisfied at the wall, i.e.,

$$u=w=0 \tag{5}$$

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The wall is deformable; therefore, from the nonflow condition we obtain

$$v = \mathrm{Re}\left\{\frac{ip_0\beta}{C_d}\exp\left[i\left(k_x x + k_z z - \beta t + \theta\right)\right]\right\} \tag{6}$$

Here p_0 is the amplitude of the pulsed pressure at the wall, and C_d is the dynamic rigidity of the elastic boundary [2]. For $C_d \to \infty$ we obtain v = 0, i.e., the boundary condition for a rigid boundary; θ is the angle of phase shift between the pressure and the displacement of the boundary, which is one of the fundamental parameters in the theory of oscillations [2].

For a passive coating, we have $0 \le \theta \le 180°$. The phase-shift angle is a function of the frequency β.

At the outer boundary of the sublayer we assume the pulsations in velocity u and w to be known

$$\begin{aligned} u &= \mathrm{Re}\left\{C_{\vartheta e}\exp\left[i\left(k_x x + k_z z - \beta t\right)\right]\right\} \\ w &= \mathrm{Re}\left\{B_{\vartheta e}\exp\left[i\left(k_x x + k_z z - \beta t\right)\right]\right\} \end{aligned} \tag{7}$$

The problem is solved under the assumption that $\partial p/\partial y = 0$ in the sublayer. Furthermore, we assume $w \sim u$, where the proportionality coefficient is constant in the sublayer, in principle, different for each harmonic and assumed known. Following the scheme of Sternberg [1]

$$w = u\,\mathrm{tg}\,\vartheta, \qquad \lambda_x = \lambda_z\,\mathrm{tg}\,\vartheta \tag{8}$$

where ϑ is the angle of skewedness of the pulsations. In this case the second boundary condition (7) is replaced by the relation

$$\mathrm{tg}\,\vartheta = B_{\vartheta e}/C_{\vartheta e}$$

In order to solve the formulated problem, it is sufficient to take into account the assumptions made about the written boundary conditions. In practice, it is necessary to solve only two of the four equations of (1) – the first and the last. The third equation is similar to the first.

The expression for the longitudinal pulsation of the velocity u, obtained by integration of the first of equations (1) with account of the two boundary conditions from (5) and (7), is similar to the equation from [1]. The square of the root-mean-square pulsation $\langle u_e^2\rangle$ is calculated as the real part of half the product of the amplitude h and the complex-conjugate quantity $\bar{h}$

$$\frac{\langle u_e^2\rangle}{{}^1\!/_2\, C_e^2} = 1 - 2e^{-\eta}\cos\eta + e^{-2\eta}, \qquad \eta = \left(\frac{\beta}{2\nu}\right)^{1/2} y \tag{9}$$

Here η is the dimensionless ordinate.

This solution differs from the Sternberg solution by only a constant. The magnitude of C_e should be determined experimentally or theoretically from the interaction of the elastic boundary with the turbulent core. For example, can use an approximate energy formulation to determine C_e. In this case, when absorption by the wall of energy of pulsations from the core is minimal, $C_e \to C$, where C is constant for the outer boundary of the sublayer near a solid wall [1].

From the last of equations (1), taking account of the boundary condition (6), we determine the transverse velocity pulsation

$$g_e = \frac{C_{\vartheta e} k_x (1 + \mathrm{tg}^2\vartheta)}{(\beta/2\nu)^{1/2}}\left[-i\eta - \frac{i-1}{2}e^{(i-1)\eta} + \frac{i-1}{2} + \frac{ip_0 e^{i\theta}}{C_{\vartheta e}}\frac{U_w(\beta/2\nu)^{1/2}}{C_d(1+\mathrm{tg}^2\vartheta)}\right] \tag{10}$$

The written value of the amplitude of the transverse pulse velocity can be represented as the sum of two components

$$g_e = g + g_1 \qquad (g_1 = ip_0 e^{i\theta}\beta/C_d) \tag{11}$$

The quantity g_1 characterizes the action of the elastic boundary on the transverse velocity pulsation, which proves to be constant in the sublayer. The component g depends on the elastic properties of the wall, and is expressed only in terms of the coefficient $C_{\vartheta e}$. With increasing rigidity of the boundary, i.e., for $C_d \to \infty$, we have $g_1 \to 0$.

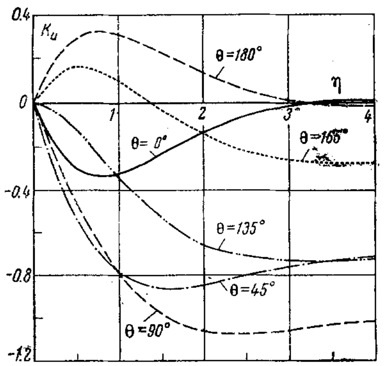

Fig. 1

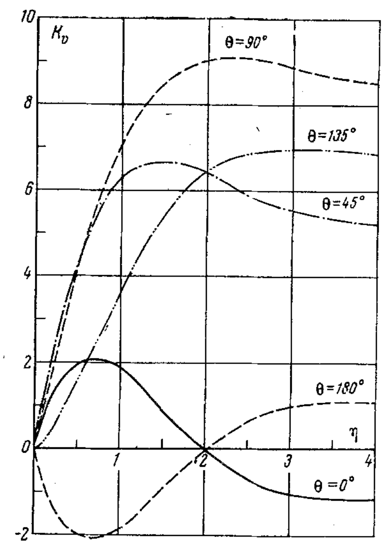

Fig. 2

We can now calculate the change in Reynolds stresses in the sublayer as a result of the action of the elastic boundary. The Reynolds stress is calculated for each individual component of the pulsation harmonic, and is determined by the correlation $\langle uv\rangle_e$:

$$\langle uv\rangle_e = {}^1/_2 \operatorname{Re}\{g_e(y)\, \bar{h}(y)\}$$

Here the overbar denotes a conjugate quantity. To calculate g_1 we must know the pulsed pressureat the elastic boundary.

In the sublayer we have $\partial p/\partial y = 0$; therefore, the pressure can be determined at its outer boundary for η_l.

To calculate p we use Eq. (13) of [1], which already takes into account the convective terms. For $\eta = \eta_l$, we have

$$p = \rho U_w \operatorname{Re}\left\{u_l\left(1 - \frac{U_l}{U_w}\right) + \frac{iv_l}{\beta}\left(\frac{dU}{dy}\right)_l\right\} \tag{12}$$

where U_l is the known velocity of the mean flow, and u_l and v_l are the pulsed velocities for η_l.

We assume that it is possible to calculate $\langle uv\rangle_e$ neglecting the change in p with change in v because of the action of the elastic boundary.

In this case the Reynolds stress in the sublayer around it is written as follows:

$$\rho\langle -uv\rangle_e = \frac{\rho C_e^2}{2}(1 + \operatorname{tg}^2\vartheta)\left(\frac{\pi f/\nu}{U_W^2}\right)^{1/2}\Big\{[1 - 2e^{-\eta}\cos\eta - 2\eta e^{-\eta}\sin\eta + e^{-2\eta}] - \\ - \rho\frac{U_w^2}{C_d}\frac{d(U_l/U_w)}{dy}K_v - \frac{2\rho U_w^2(\beta/2\nu)^{1/2}}{C_d(1 + \operatorname{tg}^2\vartheta)}\left(1 - \frac{U_l}{U_w}\right)K_u\Big\} \tag{13}$$

Here

$$K_u(\eta, \theta) = -\sin\theta + e^{-\eta_l}\sin(\eta_l + \theta) + e^{-\eta}\sin(\theta - \eta) \\ - e^{-(\eta_l+\eta)}\sin(\eta_l + \theta - \eta) \tag{14}$$

$$K_v(\eta, \theta) = -\sin\theta + e^{-\eta_l}\sin(\eta_l + \theta) - \cos\theta + e^{-\eta_l}\cos(\eta_l + \theta) + 2\eta_l\sin\theta + \\ + e^{-\eta}\sin(\theta - \eta) + e^{-\eta}\cos(\theta - \eta) - 2\eta_l e^{-\eta}\sin(\theta - \eta) - \\ - e^{-(\eta_l+\eta)}\sin(\eta_l + \theta - \eta) - e^{-(\eta_l+\eta)}\cos(\eta_l + \theta - \eta) \tag{15}$$

In Eq. (13) the term in square brackets determines the Reynolds stress near the solid boundary. The additional terms obtained by taking account of the elastic properties of the boundary can give both positive or negative contributions to the turbulent friction, depending on these properties. It is experimentally determined that the velocity of transport of turbulence $U_W \approx 0.8U_\infty$. Thus, for a very broad band of frequencies practically for the entire ordinarily considered bond, we have $U_l < U_W$.

For most of the interesting nondetached flows we have $dU/dy > 0$ over the entire boundary-layer flow.

Therefore, the Reynolds stress should decrease from the action of the elastic boundary if $K_u > 0$ and $K_v > 0$, while it should increase if $K_u < 0$ and $K_v < 0$. From the solution (9) it follows that $\eta_l \approx 5$. For convenience in calculation we set $\eta_l = 3\pi/2$. Figures 1 and 2 present the results of calculation of the dependence of the coefficients K_u and K_v on η for some values of the phase-shift angle.

An analysis of Eq. (14) shows that the term with K_u will give an increase in the Reynolds stress over practically the entire range of η for $\theta < 135°$. For $\theta > 135°$ in the range of small η we have $K_u > 0$; however, this term gives a decrease of the Reynolds stress in the sublayer in the integral sense ($\langle K_u\rangle > 0$) only for $\theta > 173°$. The quantity K_u attains its maximum absolute value for $\theta = 90°$.

An analysis of Eq. (15) shows that the term with K_v ensures a decrease in the Reynolds stress over the entire range of η for $7° < \theta < 142°$, where the maximum efficiency is attained also at $\theta = 90°$. In the

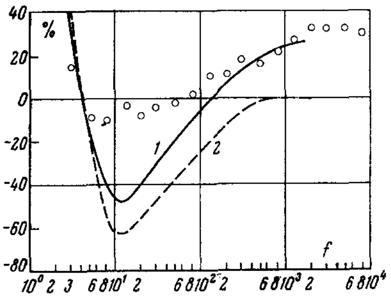

Fig. 3

integral sense this term ensures a decrease in the Reynolds stress over the entire range of phase-shift angles for a passive coating ($\langle K_v \rangle > 0$).

It is clear that since the contribution of the terms with K_u and with K_v are opposite, the attainment of a positive efficiency of action of the elastic coating on the turbulent friction is possible only in certain cases when definite conditions are satisfied. For a scheme of coating with strong damping it is important to show the necessary dependence of the phase-shift angle on frequency.

The optimal dependence can be determined, evidently, from the condition of maximum decrease of Reynolds stresses over the entire sublayer.

If l is the thickness of the sublayer, then their mean change in it (if we set $C_e = C$) is

$$\frac{1}{l}\int_0^l \frac{\langle -uv\rangle - \langle -uv\rangle_e}{{}^1/_2\, C^2}\, dy = \frac{2\rho v_d^3}{\nu C_d} G \tag{16}$$

Here v_d is the dynamic velocity,

$$G = \left(\frac{\pi f \nu}{v_d^2}\right)\left[\left(\frac{U_w}{v_d} - \frac{U_l}{v_d}\right)\langle K_u\rangle + \frac{\nu}{v_d^2}\left(\frac{dU}{dy}\right)_l \left(\frac{\pi f \nu}{v_d^2}\right)^{-1/2} \frac{1 + \mathrm{tg}^2\vartheta}{2}\langle K_v\rangle\right. \tag{17}$$

$$\langle K_u\rangle = \frac{1}{\eta_l}\int_0^{\eta_l} K_u\, d\eta, \quad \langle K_v\rangle = \frac{1}{\eta_l}\int_0^{\eta_l} K_v\, d\eta$$

Investigating Eq. (16) at the extremum, we can determine the optimal frequency characteristic. However, such a calculation would be too inaccurate, since at the present time we do not have exact information on a number of the parameters that appear in (17). Thus, the effect of ϑ can be very substantial because the term with $\langle K_v \rangle$ is directly proportional to $(1 + \tan^2\vartheta)$; however, information about the frequency dependence in the sublayer of the angle of skewness of pulsations, unfortunately, is not yet available. We also need exact information on the frequency dependence of the velocity of transport of turbulence.

Accuracy is required here because the function G proves to be a comparatively small difference between two large terms containing the factors $\langle K_u \rangle$ and $\langle K_v \rangle$.

The problem of the interaction of an elastic boundary with a viscous sublayer is solved within the framework of strong limitations; therefore, it is important to compare the results of the approximate calculation with the experimental data. It is true that there are not enough data in this area. Only Blick and co-workers [3] succeeded in measuring the characteristics of turbulence in a sublayer near an elastic boundary. They give data on the change, under the action of an elastic coating, of the spectral density of energy as a function of frequency for the dimensionless ordinate $y/\delta = 0.0033$, which corresponds to $y_+ = 9.8$ ($y_+ = v_d y/\nu$). These data are denoted by circles in Fig. 3. It is useful to compare these data with the results of calculation of the change of Reynolds stresses, since the latter are evidently proportional to the kinetic energy of turbulence. Figure 3 presents for comparison the results of calculation of the relative change of Reynolds stresses

$$\frac{\langle -uv\rangle - \langle -uv\rangle_e}{\langle -uv\rangle} 100\%$$

under the action of an elastic boundary, satisfied for $y_+ = 9.8$, $U_w = 0.8U_\infty$, $\vartheta = 45°$. The case in which $C_e = C$ is denoted by the dashed line. Since the work cited above also presents data for the change in spectral energy density in the turbulent core, we can calculate C_e. For the calculation we use the data of Fig. 20 from [3] for $y/\delta = 0.8$. Taking these data into account, we obtain results represented in Fig. 3 by the solid curve.

From a consideration of Fig. 3 we can note a definite qualitative similarity between the calculation results and the experimental data.

The quantitative divergence is not that surprising. It is evident that the convective terms in equations (1) were neglected. There are other important possible reasons, such as: ignorance of the law of variation with frequency of the skewedness of the flow, and of the transport velocity. Nevertheless, the qualitative similarity between the pictures obtained convinces us of the usefulness of the present consideration as a first step toward explaining the physics of the effect of the elastic boundary on frictional drag.

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